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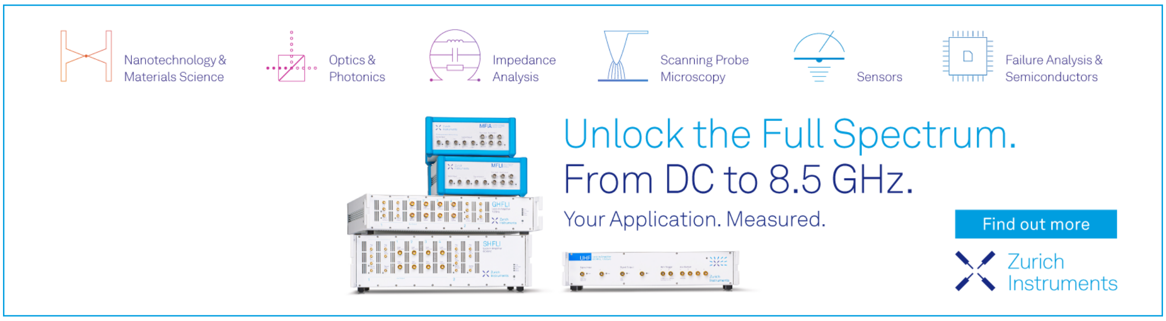

Free Vibrations of a Spherical Shell With a Pinched Edge

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Abstract. Shells as elements of machines and structures are widely used in aircraft shipbuilding and engineering. Shells as elements of machines and structures are widely used in aviation, shipbuilding and various fields of engineering and technology. In this paper the problem of free vibrations of a spherical shell with a pinched edge is considered. The equations of oscillations of a thin elastic spherical shell with a pinched edge are given. The solution is expressed through Lejandre and exponential functions. Based on the methods of special functions of mathematical physics, Mueller method and Gauss method, an algorithm for solving the problem on a computer is developed. An engineering method for determining the natural frequencies of a spherical shell with a pinched edge is proposed. Numerical results are obtained on the basis of the constructed frequency equations with complex input parameters.

INTRODUCTION

Shells as elements of machines and structures are widely used in aircraft shipbuilding and engineering. Therefore, in recent times, researchers have shown considerable interest in the issues related to the dynamic behavior of thin-walled structures that are in contact with the external environment under operating conditions [1-4]. In connection with oil and gas production, the need for storage, transportation and processing of a variety of chemical liquids, the problems of strength and resource of tank shells are very relevant. In addition, the Earth can be considered as a spherical shell with a filler. In works [5-8] frequencies and forms of free vibrations of spherical and cylindrical shells are investigated. Approximate simple formulas for calculating the frequency and determining the shape of vibrations of the considered systems are obtained by asymptotic methods, which limits the use of the obtained results, since it excludes in some important cases the possibility of qualitative analysis of the studied processes. These studies also involve great difficulties due to the need to solve transcendental systems of equations [9-11]. In [12], free oscillations of a thin-walled shell containing a compressible fluid are investigated. At some values of the system parameters, its natural frequencies of vibration were determined, and the influence of geometric and physical parameters of the system "cylindrical shell-liquid" on the free vibrations of the cylinder was investigated [13-14].

This paper studies the problem of free vibrations of a thin-walled elastic spherical shell with a pinched edge.

METHOD

Let us consider the free axisymmetric oscillations of a thin spherical shell, using for this purpose the equations of the moment theory of Liave-Kirchhoff shell calculation. Then the system of differential equations describing the oscillation process can be represented in the following form [15]:

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$$\begin{cases} \Delta U + 2U - \frac{(1-\mu)^2 R^2 \rho}{E} \frac{\partial^2}{\partial t^2} [U + (2+\mu)\omega] = 0; \\ \Delta^2 \omega + 2\Delta\omega + c^2\omega - \frac{c^2}{1-\mu} U + \frac{R^2 \rho}{E} \frac{\partial^2}{\partial t^2} [2(1-\mu^2)U + c^2\omega] = 0, \end{cases} \tag{1}$$

where

$$U = \frac{\partial u}{\partial \theta} + uctg\theta - (1+\mu)\omega \tag{2}$$

$$c^2 = 12(1-\mu^2)\frac{R^2}{\delta^2}\left[1 + \frac{\delta^2}{6R^2}\right] \tag{3}$$

$$\Delta(\dots) = \frac{\partial^2(\dots)}{\partial \theta^2} + ctg\theta \frac{\partial(\dots)}{\partial \theta} \tag{4}$$

μ - Poisson's coefficient; δ - shell thickness; ρ - specific mass; E - modulus of elasticity; R- radius of the shell; u, ω - displacements of the dome points along the tangent to the meridian and along the radius; θ - angle between the vertical and the radius-vector directed from the center of the sphere to the given point on the meridian under consideration.

Having separated the variables, having decomposed the required quantities by the forms of natural oscillations, we have the following solving equation for the *n*-th form of oscillations:

$$\Delta^3\omega_n + (4 + k_n^2)\Delta^2\omega_n + c^2\left[1 - \frac{R_n^2}{1-\mu^2}\right]\Delta\omega_n +$$
$$c^2\left[2 + \frac{1+3\mu}{1-\mu^2}k_n^2 - \frac{R_n^2}{1-\mu^2}\right]\omega_n = 0, \tag{5}$$

where ω_n - *is the* amplitude value of displacement along the radius at oscillation by the n-th form, and the function U_n is related to the displacement along the radius by the relation

$$U_n = \frac{1-\mu}{c^2 + 2(1-\mu)k_n^2}\left(\Delta^2\omega_n + 2\Delta\omega_n + c^2\left[1 - \frac{k_n^2}{1-\mu^2}\right]\omega_n\right). \tag{6}$$

In this connection the displacement u_n is determined by the solution of the differential equation

$$\frac{du_n}{d\theta} + u_n ctg\theta = U_n + (1+\mu)\omega_n \tag{7}$$

For a dome without an aperture at the pole, the solution of equation (5) can be represented in generalized spherical Lejandre functions

$$\omega_n = \sum_{i=1}^{3} A_i P_{ni}(\cos\theta) \ . \tag{8}$$

Here A_i *are the* integration constants, n_i and must satisfy the following characteristic cubic equation with respect to p_i

$$p_i^3 - (k_n^2 + 4)p_i^2 + \left[1 - \frac{k_n^2}{1-\mu^2}\right]c^2 p_i - \left[2 + \frac{(1+3\mu)k_n^2 - k_n^4}{1-\mu^2}\right]c^2 = 0 \ , \tag{9}$$

where $p_i = n_i(n_i + 1)$-is the frequency coefficient included in the resulting equation is still an unknown quantity. If we could somehow find the values of k_n, *then* p_i they would be defined as the roots of the cubic equation:

$$\begin{cases} p_1 = 2\sqrt{\frac{-p}{3}}\cos\frac{\varphi}{3} + \frac{k_n^2+4}{3}; \\ p_2 = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) + \frac{k_n^2+4}{3}; \\ p_3 = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) + \frac{k_n^2+4}{3} \end{cases} \tag{10}$$

where

$$\varphi = \arccos\left(-\frac{q}{2r}\right); r = \sqrt{-\frac{p^3}{27}} \quad , \tag{11}$$

$$p = -\frac{2(k_n^2+4)^3}{3} + \left(1 - \frac{k_n^2}{1-\mu^2}\right)c^2 \quad , \tag{12}$$

$$q = -2\frac{(k_n^2+4)^3}{27} + \frac{k_n^2+4}{3}\left[1 - \frac{k_n^2}{1-\mu^2}\right]c^2 - \left[2 + \frac{(1+3\mu)k_n^2 - k_n^4}{1-\mu^2}\right]c^2 \tag{13}$$

Checking the correctness of finding the values of p_i can be the following equations

$$\begin{cases} p_1 p_2 p_3 = \left[2 + \frac{(1+3\mu)k_n^2 - k_n^2}{1-\mu^2}\right]c^2; \\ p_1 + p_2 + p_3 = k_n^2 + 4. \end{cases} \tag{14}$$

In accordance with formula (6), let us represent the function U_n in the form

$$U_n = \sum_{i=1}^{3} \lambda_i A_i P_{ni}(\cos\theta) \ , \tag{15}$$

$$\lambda_n = \frac{p_i^2 - 2p_i + c^2\left[1 - \frac{k_n^2}{1-\mu^2}\right]}{\frac{c^2}{1-\mu} + 2k_n^2}. \qquad (16)$$

We find the displacement $u_n$as a solution of the ordinary inhomogeneous differential equation with variable coefficients (7)

$$u_n = -\sum_{i=1}^{3} \eta_i A_i P_{ni}(\cos\theta), \qquad (17)$$

where

$$\eta_i = \frac{(1-\mu^2)(p_i-2)p_i + c^2(2(1+\mu)-k_n^2)}{(1+\mu)(c^2+2(1-\mu)k_n^2)p_i}. \qquad (18)$$

It should be noted that in the process of the study the solutions due to the increase of the order of the differential equation during the formulation of the solving equations were discarded. The symbol $P_{ni}^{'}(\cos\theta)$ denotes the first derivative of the spherical function $P_{ni}(\cos\theta)$ on the coordinate θ.

All other expressions for the angle of rotation of the tangent to the meridian θ, for the normal forces along the meridian N_{1n} and parallel N_{2n} , for the moment in the meridional plane M_n and for the transverse force Q_n can be obtained using the known static, geometrical and physical equations of the moment theory of shells. At the same time, the following relation, which is valid for spherical functions and their first two derivatives, should be taken into account:

$$\Delta P_{ni}(\cos\theta) = -p_n P_n(\cos\theta) = \frac{d^2 P_n(\cos\theta)}{d\theta^2} + \frac{dP_n(\cos\theta)}{d\theta} ctg\theta, \qquad (19)$$

from where

$$\frac{d^2 P_n(\cos\theta)}{d\theta^2} = -\frac{dP_n(\cos\theta)}{d\theta} ctg\theta - p_n P_n(\cos\theta) \ . \qquad (20)$$

The last expression allows, when determining the forces in the spherical dome, to dispense with the knowledge of the second and higher derivatives. Taking into account the above mentioned, after elementary transformations, we obtain the following expressions for the rotation angle and forces corresponding to oscillations in the *n*-th principal form:

$$\vartheta_n = -\frac{1}{R}\sum_{i=1}^{3} A_i(1+\eta_i)P_{ni}^{'}(\cos\theta)\,; \qquad (21)$$

$$N_{1n} = -\frac{E\delta}{(1-\mu^2)R}\sum_{i=1}^{3} A_i\left[(1+\mu+p_i\eta_i)P_{ni}(\cos\theta) + (1+\mu)\eta_i ctg\theta P_{ni}^{'}(\cos\theta\right]; \qquad (22)$$

$$N_{2n} = -\frac{E\delta}{(1-\mu^2)R}\sum_{i=1}^{3} A_i\left[(1+\mu+\mu p_i\eta_i)P_{ni}(\cos\theta) + (1+\mu)\eta_i ctg\theta P_{ni}^{'}(\cos\theta\right]; \qquad (23)$$

$$M_n = \frac{D}{R^2}\sum_{i=1}^{3} A_i\left[(1+\mu+(1+\eta_i)p_i)P_{ni}(\cos\theta) + [\eta_i + (1-\mu+\eta_i+2\mu\eta_i)ctg\theta]P_{ni}^{'}(\cos\theta)\right]; \qquad (24)$$

$$Q_n = \frac{D}{R^3}\sum_{i=1}^{3} A_i\left[(-p_i + (1+\mu+\mu p_i - 2\mu p_i\eta_i)\cos\theta) + \left[1+\mu+p_i+p_i\eta_i - (1-\mu+\eta_i+2\mu\eta_i)\frac{1}{\sin^2\theta}\right]P_{ni}^{'}(\cos\theta)\right]; \qquad (25)$$

where D - cylindrical stiffness, equal to

$$D = \frac{E\delta^3}{12(1-\mu^2)}. \qquad (26)$$

As it is known [2], the conditions of dome edge fixation are characterized by three geometric quantities u, ω, v and three static factors -N_1, M, Q, and at rigid support three of these six quantities turn to zero.

Thus, consideration of the boundary conditions allows us to obtain three additional equations defining three values of the quantity p_i, which makes it possible to find the frequency of free oscillations ω_n, which is calculated by the formula

$$\omega_n = \frac{k_n}{R}\sqrt{\frac{(1-\mu^2)E}{\rho}}. \qquad (27)$$

Hence, the solution of the problem on the frequencies of natural vibrations is reduced to finding such a triple of values of , which satisfy the boundary conditions of the problem under consideration and which are the solution of equation (5) at some desired value of k_n . If we consider a dome with a pinched edge, then in this case there are no displacements of the edge points and rotations of the support sections in the embedding when the dome oscillates in any n-th form. The edge restraint conditions can be expressed mathematically in the following form: at $\theta = \alpha$: $u_n = 0, \omega_n = 0, \vartheta_n = 0$.

After substituting the corresponding expressions obtained above, we can write:

$$\begin{cases} \sum_{i=1}^{3} A_i P_{ni}(\cos\alpha) = 0; \\ -\sum_{i=1}^{3} A_i \eta_i P_{ni}^{'}(\cos\alpha) = 0; \\ -\sum_{i=1}^{3} A_i (1+\eta_i) P_{ni}^{'}(\cos\alpha) = 0; \end{cases} \qquad (28)$$

In defining spherical functions and their derivatives, let us represent them as infinite series:

$$P_{n1}(\cos\alpha) = 1 - p_i \sin^2\frac{\alpha}{2} + \frac{1}{2}p_i\left(\frac{p_i}{1*2}-1\right)\sin^4\frac{\alpha}{2} - \frac{1}{3}p_i\left(\frac{p_i}{1*2}-1\right)\left(\frac{2}{2*3}-1\right)\sin^6\frac{\alpha}{2} + \frac{1}{4}p_i\left(\frac{p_i}{1*2}-1\right)\left(\frac{p_i}{2*3}-1\right)\left(\frac{p_i}{3*4}-1\right)\sin^8\frac{\alpha}{2}\ldots \quad (29)$$

$$P'_{n1}(\cos\alpha) = -\frac{p_i \sin\alpha}{2}[1 - \left(\frac{p_i}{1*2}-1\right)\sin^2\frac{\alpha}{2} + \left(\frac{p_i}{1*2}-1\right)\left(\frac{2}{2*3}-1\right)\sin^4\frac{\alpha}{2} - -\left(\frac{p_i}{1*2}-1\right)\left(\frac{p_i}{2*3}-1\right)\left(\frac{p_i}{3*4}-1\right)\sin^6\frac{\alpha}{2}\ldots \quad (30)$$

It should be noted that the question of multiple roots, when the solution of equation (5) has to be represented in the form of

$$\omega_m = A_1 P_{n1}(\cos\theta) + A_2 P_{n2}(\cos\theta) + A_3 P_{n3}(\cos\theta)\int_0^6 \frac{d\theta}{P_{n2}^2(\cos\theta)\,si} \quad . \quad (31)$$

This is done intentionally, since detailed investigations have shown that the case of multiple roots can occur only at two values of k_n. One of these values lies below the first lowest frequency of free oscillations. The second value is so large that it corresponds to the values of frequencies of free oscillations of high order, which are not of interest in solving practical problems.

The most effective way to solve the problem of determining the frequencies of free oscillations is to make the cosine of the angle φ determined by formula (11) less than unity for a given k_n. Further, using the formulas (10), the values of p_n are determined according to which the values of the spherical functions $P_n(\cos\alpha)$ and their derivatives $P'_{ni}(\cos\alpha)$ are found. Substitution of the found values into the determinant (29) in general case does not lead to its equality to zero. In this connection we have to set a new value of k_n and perform the above calculations many times. If the found values are put off on the graph as functions of the value k_n and a smooth curve is plotted, the points of its intersection with the abscissa axis will give, with sufficient accuracy for practical purposes, the true values of the frequency coefficients, and hence the values of the true frequencies of free oscillations. After the calculations have been carried out, the question of the principal forms of oscillations is solved elementarily. Taking arbitrarily one of the constant coefficients in expression (8), for example, A_1, for one, we come, considering equations (28), to the following system of inhomogeneous algebraic equations:

$$\begin{cases} P_{n2}(\cos\alpha)A_2 + P_{n3}(\cos\alpha)A_3 = -P_{n1}(\cos\alpha); \\ \eta_2 P'_{n2}(\cos\alpha)A_2 + \eta_3 P'_{n3}(\cos\alpha)A_3 = -\eta_1 P'_{n1}(\cos\alpha). \end{cases} \quad (32)$$

Solving these equations leads to integration constants such as:

$$A_3 = -\frac{\eta_1 P'_{n1}(\cos\quad)P_{n2}(\cos\alpha) - \eta_2 P_{n1}(\cos\alpha)P'_{n2}(\cos\alpha)}{\eta_3 P_{n2}(\cos\alpha)P'_{n3}(\cos\alpha) - \eta_2 P'_{n2}(\cos\alpha)P_{n3}(\cos\alpha)}. \quad (33)$$

Substituting the found values of constants into expressions (8), (17) and (21), we obtain the corresponding expressions with respect to any n-th form of free oscillations.

RESULTS AND DISCUSSION

As shown by the studies performed, as well as by the solution of case studies, one of which will be discussed in detail below, the results obtained with the moment theory of shell calculation differ significantly from those given by the momentless version of the theory. The values of the spherical functions (28) and their derivatives are summarized in Table 1.

TABLE 1. Values of spherical functions and their derivatives.

P	n	τ	$P(\cos 60^0)$	$P'(\cos 60^0)$
75.2	8.17	-	-0.1066	-2.0564
-67.9	-	8.22	818.888	5834.29
-1.3	-	1.025	1.408	0.3925

At the same time, it is necessary to make some remarks concerning the use of an approximate method for determining the frequencies of natural vibrations of spherical shells, which we proposed earlier [1]. In this work, the differential equations of free oscillations of the shell were obtained and the technique of softening the boundary conditions, which consisted in keeping in a finite number of terms, was used to determine the frequencies. The scope of application of the noted approximate technique has not been established. Comparing now the exact solutions with the previously obtained approximate solutions, we can indicate that the previously recommended approximate technique determines the third and higher frequencies of free oscillations with sufficient accuracy. Determination of the frequency of natural oscillations will be carried out by selection, for this purpose we set the value of the frequency coefficient $k^2 = 2$. Let's calculate by formula (3) the value c^2:

$$c^2 = 12(1 - 0{,}25^2)20^2\left(1 + \frac{1}{20^2}\right) = 4500.$$

By formulas (11)-(13) determine $p, q, r, \cos\varphi$. Calculate y_1, y_2 and y_3 by formulas (10):

$$y_1 = 2\sqrt[3]{70445}*0.885 = 73.2;$$
$$y_2 = 2\sqrt[3]{70445}*(-0.845) = -69.9;$$
$$y_3 = 2\sqrt[3]{70445}*(-0.040) = -3.3.$$

We now find the roots of the cubic equation p_1, p_2 and p_3 :

$$p_1 = 73.2 + \frac{2+4}{3} = 75.2;$$
$$p_2 = -69.9 + \frac{2+4}{3} = -67.9;$$
$$p_3 = -3.3 + \frac{2+4}{3} = -1.3.$$

Thus, an engineering method for determining the natural frequencies of a spherical shell is proposed.

In [16-20] the motion control of mechanical systems by constraining the system by links is considered, which can be successfully applied to the control of the natural frequencies of a spherical shell with a pinched edge.

CONCLUSION

An engineering method is thus proposed to determine the natural frequencies of a spherical shell with a pinched edge. The free axisymmetric vibrations of a thin spherical dome have been considered, using for this purpose the equations of the moment theory of Liave-Kirchhoff shell calculation. Based on the methods of special functions of mathematical physics, Mueller method and Gauss method, an algorithm for solving the problem on a computer has been developed. Numerical results are obtained on the basis of the constructed frequency equations with complex input parameters.

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